Cross-intersecting families of permutations

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Abstract

For positive integers r and n with $r \leq n$, let $\mathcal{P}_{r,n}$ be the family of all sets $\{(1, y_1), (2, y_2), ..., (r, y_r)\}$ such that $y_1, y_2, ..., y_r$ are distinct elements of $[n] = \{1, 2, ..., n\}$. $\mathcal{P}_{n,n}$ describes *permutations* of $[n]$. For $r < n$, $\mathcal{P}_{r,n}$ describes *permutations* of r-element subsets of $[n]$. Families $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ of sets are said to be *cross-intersecting* if, for any distinct i and j in $[k]$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$. For any r, n and $k \geq 2$, we determine the cases in which the sum of sizes of cross-intersecting sub-families $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ of $\mathcal{P}_{r,n}$ is a maximum, hence solving a recent conjecture (suggested by the author).

1 Introduction

For a positive integer n, the set $\{1, 2, ..., n\}$ is denoted by $[n]$. The family of all r-element subsets of a set X is denoted by $\binom{X}{r}$.

If $\mathcal{F}$ is a family of sets and x is an element of the union of all sets in $\mathcal{F}$, then we call the sub-family of $\mathcal{F}$ consisting of those sets that contain x a *star of* $\mathcal{F}$ with *centre* x.

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect. Note that a star of a family is trivially intersecting.

The classical Erdős-Ko-Rado (EKR) Theorem [5] says that, if $r \leq n/2$, then an intersecting sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ has size at most $\binom{n-1}{r-1}$, i.e. the size of a star of $\binom{[n]}{r}$. If $r < n/2$ then, by the Hilton-Milner Theorem [7], $\mathcal{A}$ attains the bound if and only if $\mathcal{A}$ is a star of $\binom{[n]}{r}$. The EKR Theorem inspired a wealth of results and continues to do so; the survey papers [4, 6] are recommended.

For integers r and n with $r \leq n$, let

$$\mathcal{P}_{r,n} = \{\{(1, y_1), (2, y_2), ..., (r, y_r)\} \colon y_1, y_2, ..., y_r \text{ are distinct elements of } [n]\}.$$

$\mathcal{P}_{n,n}$ describes permutations of the set $[n]$ because a permutation $y_1 y_2 ... y_n$ of $[n]$ corresponds uniquely to the set $\{(1, y_1), (2, y_2), ..., (n, y_n)\}$ in $\mathcal{P}_{n,n}$. $\mathcal{P}_{r,n}$ describes permutations of r-subsets of $[n]$ because a permutation $y_1 y_2 ... y_r$ of an r-subset of $[n]$ corresponds uniquely to the set $\{(1,y_1),(2,y_2),...,(r,y_r)\}$ in $\mathcal{P}_{r,n}$. If two sets $\{(1,y_1),(2,y_2),...,(r,y_r)\}$ and $\{(1,z_1),(2,z_2),...,(r,z_r)\}$ in $\mathcal{P}_{r,n}$ intersect, then $y_i = z_i$ for some $i \in [r]$, and this is exactly what we mean by saying that the permutations $y_1y_2...y_r$ and $z_1z_2...z_r$ (of two r-subsets of $[n]$) intersect.

Deza and Frankl [3] proved an analogue of the EKR Theorem for permutations, and Cameron and Ku [2] showed that only the stars of $\mathcal{P}_{n,n}$ are intersecting sub-families of $\mathcal{P}_{n,n}$ of maximum size. A more general result was obtained by Larose and Malvenuto [8, Theorem 5.1].

Theorem 1.1 ([8]) *If $\mathcal{A}$ is an intersecting sub-family of $\mathcal{P}_{r,n}$, then $|\mathcal{A}| \leq \frac{(n-1)!}{(n-r)!}$, and equality holds if and only if $\mathcal{A}$ is a star of $\mathcal{P}_{r,n}$.*

Families $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are said to be *cross-intersecting* if, for any $i,j \in [k]$ such that $i \neq j$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$. The study of cross-intersecting families is related to that of intersecting families, and results for cross-intersecting families are generally stronger than results for intersecting families; this will become clear from the rest of the paper.

A special case of [1, Theorem 2.4] is the following cross-intersection result for permutations.

Theorem 1.2 ([1]) *If $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are cross-intersecting sub-families of $\mathcal{P}_{r,n}$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \frac{n!}{(n-r)!} & \text{if } k \leq n; \\ k\frac{(n-1)!}{(n-r)!} & \text{if } k \geq n, \end{cases}$$

and the bound is sharp.

In the same paper it is conjectured that the extremal structures for the above result are as stated below.

Conjecture 1.3 ([1]) *If $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are cross-intersecting sub-families of $\mathcal{P}_{r,n}$ and $\sum_{i=1}^{k} |\mathcal{A}_i|$ is a maximum, then one of the following holds:*
(i) $k < n$ and, for some $i \in [k]$, $\mathcal{A}_i = \mathcal{P}_{r,n}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [k]\backslash\{i\}$;
(ii) $k > n$ and $\mathcal{A}_1 = \mathcal{A}_2 = ... = \mathcal{A}_k = \{A \in \mathcal{P}_{r,n} \colon (x,y) \in A\}$ for some $(x,y) \in [r] \times [n]$;
(iii) $k = n$ and $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are as in (i) or (ii).

In this paper we show that this conjecture is true except that in the special case $2 \leq k \leq 3 = r = n$ there are other optimal configurations, which we will determine.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the sub-families of $\mathcal{P}_{3,3}$ given by

$$\mathcal{T}_1 = \{\{(1,(1+i) \bmod^* 3),(2,(2+i) \bmod^* 3),(3,(3+i) \bmod^* 3)\} \colon i = 0,1,2\},$$
$$\mathcal{T}_2 = \{\{(1,(1+i) \bmod^* 3),(2,(3+i) \bmod^* 3),(3,(2+i) \bmod^* 3)\} \colon i = 0,1,2\},$$

where 'mod*' represents the usual *modulo operation* with the exception that, for any two integers a and b, ba mod* a is a instead of 0.

Theorem 1.4 *If $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are cross-intersecting sub-families of $\mathcal{P}_{r,n}$, then $\sum_{i=1}^{k} |\mathcal{A}_i|$ is a maximum if and only if either one of (i), (ii), (iii) of Conjecture 1.3 holds or one of the following holds:*
(iv) $2 \leq k \leq 3 = r = n$, $\mathcal{A}_j = \mathcal{T}_1$ and $\mathcal{A}_l = \mathcal{T}_2$ for some $j \in [k]$ and $l \in [k] \backslash \{j\}$, and if $k = 3$ then $\mathcal{A}_p = \emptyset$ for $p \in [k] \backslash \{j, l\}$;
(v) $k = r = n = 3$, $\mathcal{A}_j = \mathcal{T}_i \cup \{T\}$ for some $j \in [3]$, $i \in [2]$, $T \in \mathcal{T}_{3-i}$, and $\mathcal{A}_l = \{T\}$ for each $l \in [3] \backslash \{j\}$.

Note that this result generalises Theorem 1.1: take $\mathcal{A}$ to be any intersecting sub-family of $\mathcal{P}_{r,n}$, and take $\mathcal{A}_1 = \mathcal{A}_2 = ... = \mathcal{A}_k = \mathcal{A}$ and $k > n$ in the above result.

For any family $\mathcal{A}$, let $\mathcal{A}^*$ be the sub-family of $\mathcal{A}$ consisting of those sets in $\mathcal{A}$ that intersect each set in $\mathcal{A}$, and let $\mathcal{A}' = \mathcal{A} \backslash \mathcal{A}^*$. So $\mathcal{A}'$ consists of those sets in $\mathcal{A}$ that do not intersect all sets in $\mathcal{A}$.

We arrive at Theorem 1.4 by proving the following extension of Theorem 1.1.

Theorem 1.5 *For any $\mathcal{A} \subseteq \mathcal{P}_{r,n}$,*

$$|\mathcal{A}^*| + \frac{1}{n}|\mathcal{A}'| \leq \frac{(n-1)!}{(n-r)!},$$

and equality holds if and only if one of the following holds:
(i) $\mathcal{A}' = \mathcal{P}_{r,n}$ and $\mathcal{A}^* = \emptyset$;
(ii) $\mathcal{A}' = \emptyset$ and $\mathcal{A}^*$ is a star of $\mathcal{P}_{r,n}$;
(iii) $r = n = 3$, $\mathcal{A}' = \mathcal{T}_i$ for some $i \in [2]$, and $\mathcal{A}^* = \{T\}$ for some $T \in \mathcal{T}_{3-i}$.

2 Proofs

Proof of Theorem 1.5. The cases $n = 1$ and $n = 2$ are trivial, so we assume $n \geq 3$.

For an integer z, let $\theta_z : \mathcal{P}_{r,n} \to \mathcal{P}_{r,n}$ be the *translation operation* defined by

$$\theta_z(A) = \{(x, (y + z) \bmod^* n) \colon (x, y) \in A\}.$$

Let $\mathcal{S}$ be the star $\{P \in \mathcal{P}_{r,n} \colon (1,1) \in P\}$. For any $S \in \mathcal{S}$, let $\mathcal{P}_S = \{S, \theta_1(S), ..., \theta_{n-1}(S)\}$ and call $\mathcal{P}_S$ an *orbit*. Clearly $\mathcal{P}_{r,n}$ is a disjoint union of the $\frac{(n-1)!}{(n-r)!}$ orbits. For any $A^* \in \mathcal{A}^*$ and $z \in [n-1]$ we have $\theta_z(A^*) \cap A^* = \emptyset$ and hence $\theta_z(A^*) \notin \mathcal{A}$; in other words, the orbit containing a set $A^* \in \mathcal{A}^*$ does not contain other sets in $\mathcal{A}$. It follows that $|\mathcal{A}^*| + |\mathcal{A}'| \leq |\mathcal{P}_{r,n}| - (n-1)|\mathcal{A}^*|$. So $n|\mathcal{A}^*| + |\mathcal{A}'| \leq \frac{n!}{(n-r)!}$ and hence

$$|\mathcal{A}^*| + \frac{1}{n}|\mathcal{A}'| \leq \frac{(n-1)!}{(n-r)!}$$

as required.

It is easy to check that the bound is attained if one of (i), (ii) and (iii) holds. We now prove the converse. So suppose the bound is attained. It is clear from the above that we

then have that, for any orbit $\mathcal{P}_S$ ($S \in \mathcal{S}$), exactly one of the following holds:
(a) the whole orbit is contained in $\mathcal{A}'$;
(b) the orbit contains exactly one set in $\mathcal{A}^*$ and no sets in $\mathcal{A}'$.
If $\mathcal{A}^* = \emptyset$ then $|\mathcal{A}'| = \frac{n!}{(n-r)!}$ and hence $\mathcal{A}' = \mathcal{P}_{r,n}$. If $\mathcal{A}' = \emptyset$ then $|\mathcal{A}^*| = \frac{(n-1)!}{(n-r)!}$ and hence, by Theorem 1.1, $\mathcal{A}^*$ is a star of $\mathcal{P}_{r,n}$.

Now suppose $\mathcal{A}^* \neq \emptyset$ and $\mathcal{A}' \neq \emptyset$. Let $A^* \in \mathcal{A}^*$, and let A_1 be a set $\{(1, y_1), (2, y_2), ..., (r, y_r)\}$ in $\mathcal{A}'$. So $A^* \cap A_1 \neq \emptyset$; we may assume $(1, y_1) \in A^* \cap A_1$. For each $z \in [n-1]$, let $A_{z+1} = \theta_z(A_1)$. Since the sets $A_1, A_2, ..., A_n$ constitute an orbit (and $A_1 \in \mathcal{A}'$), (a) tells us that these sets are all in $\mathcal{A}'$.

Suppose $r < n$. Then, since the sets $A_1, A_2, ..., A_n$ are disjoint, no set of size r can intersect each of these sets. But this gives us $\mathcal{A}^* = \emptyset$, a contradiction. Therefore $r = n$.

Suppose $n \geq 4$. Let $B_1 = (A_1 \backslash \{(1, y_1), (r, y_r)\}) \cup \{(1, y_r), (r, y_1)\}$. Clearly $B_1 \in \mathcal{P}_{n,n}$. Since $n \geq 4$, $|B_1 \cap A_1| \geq 2$. Therefore, since the sets $A_1, A_2, ..., A_n$ are disjoint (and B_1 contains n elements), we have $B_1 \cap A_j = \emptyset$ for some $j \in [n]$, and hence $B_1 \notin \mathcal{A}^*$. Similarly, for any $z \in [n-1]$ we have $\theta_z(B_1) \cap A_l = \emptyset$ for some $l \in [n]$, because $|\theta_z(B_1) \cap A_{z+1}| = |B_1 \cap A_1| \geq 2$. So the orbit containing B_1 has no sets in $\mathcal{A}^*$, and hence (a) tells us that this orbit is contained in $\mathcal{A}'$. Now A^* intersects each of the sets $A_1, A_2, ..., A_n$. Since $A_1, A_2, ..., A_n$ are disjoint, it follows that $|A^* \cap A_i| = 1$ for all $i \in [n]$. Since $(1, y_1) \in A^* \cap A_1$, we obtain $A^* \cap A_1 = \{(1, y_1)\}$. Having $(1, y_1) \in A^*$ means that $(1, y_r), (r, y_1) \notin A^*$ (simply because $A^* \in \mathcal{P}_{n,n}$), and hence we get $A^* \cap B_1 = \emptyset$, which contradicts $B_1 \in \mathcal{A}'$.

Therefore $n = 3$. So there are only two orbits $\mathcal{P}_{S_1} = \mathcal{T}_1$ and $\mathcal{P}_{S_2} = \mathcal{T}_2$, where $S_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $S_2 = \{(1, 1), (2, 3), (3, 2)\}$. Since $\mathcal{A}' \neq \emptyset$ and $\mathcal{A}^* \neq \emptyset$, it follows by (a) and (b) that one orbit is contained in $\mathcal{A}'$ and the other orbit contains exactly one set in $\mathcal{A}^*$ and no sets in $\mathcal{A}'$. This means that $\mathcal{A}' = \mathcal{T}_i$ for some $i \in [2]$ and $\mathcal{A}^* = \{T\}$ for some $T \in \mathcal{T}_j$, $j \in [2]$. Now j cannot be i as $T \in \mathcal{A}^*$ and the sets in $\mathcal{T}_i$ are disjoint. $\Box$

Lemma 2.1 *Let $r \leq n$ such that $r < n$ if $n = 3$. Suppose $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{r,n}$ such that, for any $A \in \mathcal{F}$ and $B \in \{P \in \mathcal{P}_{r,n} \colon A \cap P = \emptyset\}$, $B \in \mathcal{F}$. Then $\mathcal{F} = \mathcal{P}_{r,n}$.*

Proof. The result is trivial for $n \leq 2$, so we assume $n \geq 3$. Let $\theta_z : \mathcal{P}_{r,n} \to \mathcal{P}_{r,n}$ be defined as in the proof of Theorem 1.5. Let F_1 be a set in $\mathcal{F}$. For each $z \in [n-1]$, let $F_{z+1} = \theta_z(F_1)$. The sets $F_1, F_2, ..., F_n$ are disjoint, and hence the conditions of the lemma tell us that they are all in $\mathcal{F}$.

Consider first $r < n$. Since a set P in $\mathcal{P}_{r,n}$ can intersect at most r disjoint sets, there exists $j \in [n]$ such that $P \cap F_j = \emptyset$, meaning that $P \in \mathcal{F}$. This proves the result for $r < n$.

Now consider $r = n$; so $n \geq 4$ (by the condition of the lemma on r and n). Let P be an arbitrary set in $\mathcal{P}_{n,n}$. If $P \cap F_1 = \emptyset$ then it is immediate that $P \in \mathcal{F}$. Suppose $|P \cap F_1| \geq 2$. Therefore, since $F_1, F_2, ..., F_n$ are n disjoint sets in $\mathcal{F}$, we have $P \cap F_j = \emptyset$ for some $j \in [n]$, and hence $P \in \mathcal{F}$. Finally, suppose $|P \cap F_1| = 1$. We have $F_1 = \{(1, y_1), (2, y_2), ..., (n, y_n)\}$ for some $y_1, y_2, ..., y_n$ such that $\{y_1, y_2, ..., y_n\} = [n]$, and hence $P \cap F_1 = \{(l, y_l)\}$ for some $l \in [n]$. Let $m \in [n] \backslash \{l\}$, and let $Q = (F_1 \backslash \{(l, y_l), (m, y_m)\}) \cup \{(l, y_m), (m, y_l)\}$. Clearly $Q \in \mathcal{P}_{n,n}$. Since $n \geq 4$, we have $|Q \cap F_1| \geq 2$ and hence $Q \in \mathcal{F}$. Now $(l, y_l) \in P$ implies $(l, y_m), (m, y_l) \notin P$ (as $P \in \mathcal{P}_{n,n}$), and hence $P \cap Q = \emptyset$. So $P \in \mathcal{F}$ as $Q \in \mathcal{F}$. $\Box$

Proof of Theorem 1.4. It is easy to check that the bound in Theorem 1.2 (which we shall prove again in a different way) is attained in the cases stated in the theorem we have set out to prove. We now show that $\sum_{i=1}^{k}|\mathcal{A}_i|$ is a maximum only in those cases.

Let $\mathcal{A} = \bigcup_{i=1}^{k}\mathcal{A}_i$. Clearly $\mathcal{A}^* = \bigcup_{i=1}^{k}\mathcal{A}_i^*$ and $\mathcal{A}' = \bigcup_{i=1}^{k}\mathcal{A}_i'$. Suppose $\mathcal{A}_i' \cap \mathcal{A}_j' \neq \emptyset$, $i \neq j$. Let $A \in \mathcal{A}_i' \cap \mathcal{A}_j'$. Then there exists $A_i \in \mathcal{A}_i'$ such that $A \cap A_i = \emptyset$, which is a contradiction because $A \in \mathcal{A}_j$. So $\mathcal{A}_i' \cap \mathcal{A}_j' = \emptyset$ for $i \neq j$, and hence $|\mathcal{A}'| = \sum_{i=1}^{k}|\mathcal{A}_i'|$. Applying Theorem 1.5, we therefore get

$$\sum_{i=1}^{k}|\mathcal{A}_i| = \sum_{i=1}^{k}|\mathcal{A}_i'| + \sum_{i=1}^{k}|\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| = n\left(\frac{1}{n}|\mathcal{A}'| + |\mathcal{A}^*| + \frac{k-n}{n}|\mathcal{A}^*|\right)$$
$$\leq n\left(\frac{(n-1)!}{(n-r)!} + \frac{k-n}{n}|\mathcal{A}^*|\right) = \frac{n!}{(n-r)!} + (k-n)|\mathcal{A}^*|. \tag{1}$$

Suppose $k < n$. Then $\sum_{i=1}^{k}|\mathcal{A}_i| \leq \frac{n!}{(n-r)!}$, and equality holds if and only if $\mathcal{A}^* = \emptyset$ and $\mathcal{A} = \mathcal{A}' = \mathcal{P}_{r,n}$. Now suppose $\mathcal{A} = \mathcal{P}_{r,n}$, and let $i \in [k]$ such that $\mathcal{A}_i \neq \emptyset$. Together with the cross-intersection condition, this implies that, if $A \in \mathcal{A}_i$ and B is a set in $\mathcal{P}_{r,n}$ that does not intersect A, then B has to be in $\mathcal{A}_i$. Suppose that $n \neq 3$ or $r < n = 3$. Then the conditions of Lemma 2.1 hold for $\mathcal{A}_i$, and hence $\mathcal{A}_i = \mathcal{P}_{r,n}$. Due to the cross-intersection condition, this forces any other family $\mathcal{A}_j$ to be empty (hence (i) holds). Now suppose $r = n = 3$. Then $\mathcal{P}_{r,n}$ is the disjoint union of $\mathcal{T}_1$ and $\mathcal{T}_2$. Since $\mathcal{A} = \mathcal{P}_{r,n}$ and each of $\mathcal{T}_1$ and $\mathcal{T}_2$ consists of disjoint sets, for some $j, l \in [k]$ we have $\mathcal{T}_1 \subseteq \mathcal{A}_j$ and $\mathcal{T}_2 \subseteq \mathcal{A}_l$. If $j = l$ then $\mathcal{A}_j = \mathcal{P}_{r,n}$, which forces any other family $\mathcal{A}_p$ to be empty (hence (i) holds). Suppose $j \neq l$. Then $\mathcal{A}_j$ cannot contain sets in $\mathcal{P}_{r,n} \backslash \mathcal{T}_1 = \mathcal{T}_2$ as $\mathcal{T}_2 \subseteq \mathcal{A}_l$ and the sets in $\mathcal{T}_2$ are disjoint. So $\mathcal{A}_j = \mathcal{T}_1$ and similarly $\mathcal{A}_l = \mathcal{T}_2$ (hence (iv) holds).

Next, suppose $k > n$. Then, by (1) and the bound in Theorem 1.5,

$$\sum_{i=1}^{k}|\mathcal{A}_i| \leq \frac{n!}{(n-r)!} + (k-n)\frac{(n-1)!}{(n-r)!} = k\frac{(n-1)!}{(n-r)!}$$

with equality if and only if $\mathcal{A}_1^* = \mathcal{A}_2^* = ... = \mathcal{A}_k^* = \mathcal{A}^*$ and $|\mathcal{A}^*| = \frac{(n-1)!}{(n-r)!} = |\mathcal{A}|$. Now Theorem 1.1 tells us that, if $|\mathcal{A}^*| = \frac{(n-1)!}{(n-r)!}$, then $\mathcal{A}^*$ is a star of $\mathcal{P}_{r,n}$. Thus (ii) holds if $\sum_{i=1}^{k}|\mathcal{A}_i|$ is a maximum.

Finally, suppose $k = n$. Then, by (1),

$$\frac{1}{n}\sum_{i=1}^{k}|\mathcal{A}_i| \leq |\mathcal{A}^*| + \frac{1}{n}|\mathcal{A}'| \leq \frac{(n-1)!}{(n-r)!}. \tag{2}$$

It remains to show that the inequalities in (2) are equalities only if one of (iii), (iv) and (v) holds. If $\mathcal{A}^* = \emptyset$ then, by the argument for the case $k < n$, $\sum_{i=1}^{k}|\mathcal{A}_i|$ is a maximum only if either $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are as in (i) or (iv) holds. It is immediate from Theorem 1.5 that, if $\mathcal{A}^* \neq \emptyset$ and $r, n, \mathcal{A}^*, \mathcal{A}'$ are not as in Theorem 1.5(iii), then $\mathcal{A}^*$ is as in the case $k > n$. Now suppose $r, n, \mathcal{A}^*, \mathcal{A}'$ are as in Theorem 1.5(iii), i.e. $r = n = 3$, $\mathcal{A}' = \mathcal{T}_i$ for some $i \in [2]$, and

$\mathcal{A}^* = \{T\}$ for some $T \in \mathcal{T}_{3-i}$. We have $k = 3$ as we are considering $k = n$. Since the sets in $\mathcal{T}_i$ are disjoint, we must have $\mathcal{T}_i = \mathcal{A}'_j$ for some $j \in [3]$ and $\mathcal{A}'_l = \emptyset$ for any $l \in [3]\backslash\{j\}$. Together with $\mathcal{A}^* = \{T\}$, this gives us $\mathcal{A}_j \subseteq \mathcal{T}_i \cup \{T\}$ and $\mathcal{A}_l \subseteq \{T\}$ for $l \in [3]\backslash\{j\}$. For us to have equalities in (2) for this case, we must actually have $\mathcal{A}_j = \mathcal{T}_i \cup \{T\}$ and $\mathcal{A}_l = \{T\}$ for $l \in [3]\backslash\{j\}$ (in which case (v) holds). $\square$

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